Rebuttal of Donnelly's paper on the spectral gap

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The spectral gap conjecture of M. van den Berg [2, formula (65)] asserts that

$$\lambda_2 - \lambda_1 \geq 3\pi^2$$

for all convex euclidean domains of diameter 1, where λ_1 and λ_2 denote the first two eigenvalues of the Dirichlet Laplacian. Notice that equality holds for the 1-dimensional unit interval, which can be regarded also as a degenerate n-dimensional rectangular box.

The gap estimate is conjectured to hold more generally for Schrödinger operators with convex potentials, under Dirichlet boundary conditions; see the work of S.-T. Yau and collaborators [9, 11]. This Schrödinger gap conjecture was proved some time ago in 1 dimension by R. Lavine [8], and more recently in all dimensions by B. Andrews and J. Clutterbuck [1].

The proof in this journal by H. Donnelly [3] of the original gap conjecture in 2 dimensions (for the Dirichlet Laplacian with zero potential) is not correct. The Editors of Mathematische Zeitschrift have asked us to describe the flaws in the proof, in order to clarify the state of the literature.

Donnelly's approach to the problem is a natural one: first perform a shape optimization to rule out a non-degenerate minimizing domain, and then analyze the spectral gap for a sequence of domains degenerating to an interval, with the help of results by D. Jerison [5]. (For some history on this approach, and on the gap conjecture more generally, see the report on the AIM meeting "Low Eigenvalues of Laplace and Schrödinger Operators" [10], especially page 12 of the open problems list.)

The error lies in the proof of the shape optimization step, as we now explain. Donnelly wishes to prove that no minimizing domain can exist for

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the problem:

$$\min\{\lambda_2 - \lambda_1 : \Omega \subset \mathbb{R}^2 \text{ is convex and open with diameter } 1\}.$$

He supposes that a minimizing domain Ω does exist, and aims for a contradiction by deforming the domain under the flow of a vector field V. He correctly observes that by the Hadamard formula for the shape derivative of an eigenvalue [3, Section 2], "for any deformation preserving convexity and diameter normalization, we have the equality"

$$\int_{\partial\Omega} \left(\frac{\partial u_1}{\partial n}\right)^2 V \cdot n \, d\sigma = \int_{\partial\Omega} \left(\frac{\partial u_2}{\partial n}\right)^2 V \cdot n \, d\sigma. \tag{1}$$

(Here u_1 and u_2 denote the first two eigenfunctions of the Dirichlet Laplacian on Ω.) The derivation of this first-variation condition (1) requires the deformed domain to remain convex and have diameter 1 under both the forwards and backwards deformation of Ω.

The next sentence of the paper incorrectly deduces that $(\frac{\partial u_1}{\partial n})^2 = (\frac{\partial u_2}{\partial n})^2$ on the boundary. (In fact slightly more is claimed, that $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$, but that discrepancy is unimportant here.) The deduction is incorrect because the class of deformations that preserve convexity and the diameter normalization might not be sufficiently numerous to enable the values of $V \cdot n$ to cover a dense subset of $L^2(\partial\Omega)$. For example, the boundary of Ω might contain a curve that is not strictly convex, such as a straight line segment, and this curve would support no local perturbations that preserve convexity of the domain under both the forwards and backwards flow.

Indeed, in shape optimization problems with convexity constraints, optimality conditions generally hold only on strictly convex parts of the boundary. See, for example, some of the methods developed in the literature for dealing with convexity constraints [4, Theorem 4.2.2], [6, Section 7], and [7].

The diameter constraint is also problematic for Donnelly's argument. Nowhere does his paper explain how one should ensure that the perturbing flow fixes the diameter. The problem is serious. For example, if Ω is a domain of constant width such as a disk or a Reuleaux triangle, then most perturbing flows will change the diameter.

A final reason why we believe Donnelly's approach to be beyond repair is that if his argument were correct, then it would apply equally well to the modified gap functional $2\lambda_2 - \lambda_1$, showing that this functional cannot have a minimizing domain (among convex domains of diameter 1) and hence must be minimized by some degenerating sequence of domains. That is impossible, because $2\lambda_2 - \lambda_1 \geq \lambda_1 \to \infty$ as the sequence of domains degenerates to an interval.

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